

Block Voter Model

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Abstract. We introduce and study the block voter model with noise on two-dimensional square lattices using Monte Carlo simulations and finite-size scaling techniques. The model is defined by an outflow dynamics where a central set of N_{PCS} spins, here denoted by persuasive cluster spins (PCS), tries to influence the opinion of their neighbouring counterparts. We consider the collective behaviour of the entire system with varying PCS size. When $N_{PCS} > 2$, the system exhibits an order-disorder phase transition at a critical noise parameter q_c which is a monotonically increasing function of the size of the persuasive cluster. We conclude that how large the PCS is more power of persuasion it has. It also seems that the resulting critical behaviour is Ising-like independent of the range of the interactions.

1. Introduction

In nonequilibrium dynamical systems there are usually no energy functions and their time evolutions are defined by dynamical rules. These rules can be divided into two groups, namely, inflow and outflow dynamics[1]. For inflow dynamics the center spin is influenced by its nearest neighbours. A well-known example of such dynamics, in nonequilibrium systems, is the majority-vote model in social sciences [2, 3, 4]. On the contrary, for outflow dynamics the information flows from the center spin (or cluster of spins) to the neighbourhood. The Sznajd model [5], which was introduced to describe opinion formation in social systems, falls on this category. It is based on the fundamental social phenomenon called social validation [6]. Figure 1 illustrates the flow of influence in a typical outflow dynamics where the central spins (circle) try to influence the opinion of their neighbouring counterparts (rectangles). We shall nominate the central set of spins by persuasive cluster spins (PCS), and we aim in this paper to consider the collective behaviour of the entire system with varying PCS size. In short, the PCS will try to influence the neighbourhood and we might expect how large it is more power of persuasion it would have.

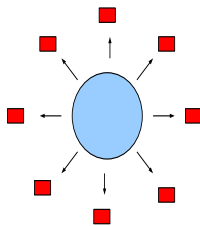


Figure 1. Schematic representation of the outflow dynamics showing the central cluster of persuasive spins (PCS) and adjacent spins (squares).

The present block voter model introduces long-range environment-behaviour interactions in the system by considering outflow dynamics. The range of the interaction is defined by the number of spins N_{PCS} inside the persuasive cluster. In principle the size of PCS can grow until to infinity (infinite range, mean-field limit). However, in the limit $N_{PCS} \rightarrow \infty$, in which every spin has the same strength of interaction with each other spin on the lattice, the system's behaviour should be predict by the mean-field theory with classical critical exponents, namely, $\beta = 1/2$, $\gamma = 1$, and $\nu = 1/2$ [7, 8, 9]. Here we will not consider this limit and restrict the simulations to finite systems taking into account the relative sizes of PCS and the whole system, which limit the number of spins in the persuasive cluster.

The remainder of the paper is organized in the following way: In Sec. 2 we introduce the model and describe the methodology used in the simulations. In Sec. 3 we present a discussion of our numerical results along with a finite-size scaling analysis of the relevant

quantities, and we conclude in Sec. 4.

2. Block Voter Model

The model system consists of a set of N two-state spin variables σ_i , associated with the i th vertex of a regular square lattice of linear size $L = \sqrt{N}$. Each spin can have two possible values $\sigma_i = \pm 1$ corresponding to the two opposite opinions in a referendum. The system's evolution starts from a completely ordered state with all spins point to the same direction. At each time step, a block of spins randomly chosen tries to influence the spins in the neighbourhood passing to them the opinion of the majority of its components. Moreover, all spins have some resistance for accepting such outflow influence. So, independently, each spin adjacent of the PCS agrees with the PCS majority state with probability p_{agree} and adopt the opposite state with probability $q = 1 - p_{agree}$. In terms of the noise parameter q , the rate of flipping for each adjacent spin is given by

$$w(\sigma_i) = \frac{1}{2} \left[1 - (1 - 2q) \sigma_i S \left(\sum_{\delta=1}^{N_{pcs}} \sigma_{i+\delta} \right) \right], \quad (1)$$

where the summation is over all N_{PCS} sites that make up the persuasive cluster, and $S(x) = \text{sgn}(x)$ if $x \neq 0$ and $S(0) = 0$. From equation 1, we notice that the block voter model shares features with the majority-vote model [3, 4]; e.g., both models exhibit up-down symmetry and are endowed with spontaneous broken symmetry on the parameter q , having then an order-disorder phase transition. However the majority-vote model is a kind of inflow dynamics which takes into account only the opinion of the neighbouring spins of a selected node i . Here we consider the majority opinion of a block of spins of size N_{PCS} with influence upon its neighbourhood. The block model is a kind of outflow dynamics defined in terms of two control parameters: q and N_{PCS} .

To study the effect of the noise parameter q and the range of interaction N_{PCS} on the phase diagram and critical behaviour of the block voter model, we consider the magnetization M_L , the susceptibility χ_L , and the Binder's fourth-order cumulant U_L [10], which are defined by:

$$M_L(q) = \left\langle \left\langle \left| \frac{1}{N} \sum_{i=1}^N \sigma_i \right| \right\rangle_{time} \right\rangle_{sample} \quad (2)$$

$$\chi_L(q) = N \left[\left\langle \langle m^2 \rangle_{time} \right\rangle - \langle m \rangle_{time}^2 \right]_{sample} \quad (3)$$

$$U_L(q) = 1 - \left\langle \frac{\langle m^4 \rangle_{time}}{3 \langle m^2 \rangle_{time}} \right\rangle_{sample} \quad (4)$$

where $N = L^2$ is the number of spins in the system. The symbols $\langle \dots \rangle_{time}$ and $\langle \dots \rangle_{sample}$, respectively, denote time averages taken in the stationary state and configurational averages taken over several samples.

We have performed Monte Carlo simulations on regular square lattices of sizes $L = 100, 120, 130, 140, 150, 160$, and 180 , for values of N_{PCS} satisfying the following constraint between the size of the persuasive cluster and the whole system: $\frac{N_{PSC}}{N} \leq 1\%$. Time is measured in Monte Carlo step (MCS), and 1 MCS corresponds to N attempts of changing the states of the spins. So, for a given PCS we consider two procedures: For *asynchronous update* we choose randomly a spin adjacent to the PCS and try to flip it with the probability given by equation (1). By repeating this procedure N times, we have accomplished one MCS. In the case of *synchronous update*, however, considering that for a given N_{PCS} we have N_{adj} adjacent spins, we can update simultaneously all theses spins with the same probability [equation (1)]. One MCS is accomplished after repeating this procedure $\overline{N} = \frac{N}{N_{adj}}$ times. For each kind of update, we waited 10^4 MCS to make the system reach the steady state, and the time averages were estimated from the next 10^4 MCS. In the critical region larger runs were needed, and we used 3×10^4 MCS to reach the steady state and 20×10^4 MCS for time averages. For all set of parameters, at least 100 independent runs (samples) were considered in the calculation of the configurational averages.

3. Results and Discussion

Figure 2 shows the dependence of the order parameter M_L and the susceptibility χ_L on the noise parameter, obtained from Monte Carlo simulations on square lattices with $L = 140$ ($N = 19600$) and several sizes of the persuasive cluster spin, namely, $N_{PSC} = 4, 9, 16, 25, 36, 49, 64$. In Fig. 2(a) each curve for M_L , for a given value of L and N_{PSC} , suggests that there exists a phase transition from an ordered to a disordered state, characterized by a spontaneous broken symmetry at a particular value of the noise parameter, namely, $q = q_c$. In the thermodynamic limit ($N \rightarrow \infty$), we should expect that below the critical noise q_c the system has a nonzero magnetization, whereas the magnetization vanishes for $q \geq q_c$. For finite systems, the value of q where each curve for χ_L in Fig. 2(b) has a maximum is identified as $q_c(L)$ for the corresponding N_{PSC} . We note that the transition occurs at a value of the critical noise parameter which is an increasing function of the size of the persuasive cluster.

For each N_{PSC} , we can obtain the critical value q_c by calculating the Binder's fourth-order magnetization cumulant $U_L(q)$ (Equation 4) as a function of the noise parameter q , for several lattice sizes L . For sufficiently large systems, these curves intercept each other in a single point $U^*(q_c)$. The value of q where occurs the intersection equals the critical noise q_c , which is not biased by any assumption about critical exponents since, by construction, the Binder's cumulant presents zero anomalous dimension[10, 11]. In Figure 3 we plot the reduced fourth-order Binder's cumulant for lattice sizes $L = 100, 120, 140, 160$, and 180 and four different values of N_{PSC} . From each set of curves, we obtain the critical noise parameter q_c as well as the critical value $U^*(q_c)$. As we can notice, there exists a strong dependence between the critical noise and the size of the persuasive cluster, i.e., when N_{PSC} increases the critical noise

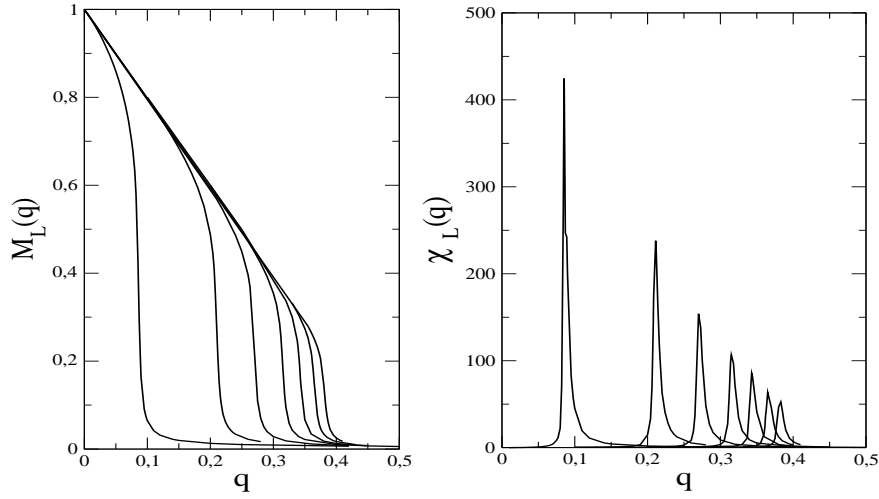


Figure 2. Magnetization and susceptibility as functions of the noise parameter q , for $L = 140$ and values of $N_{PCS} = 4, 9, 16, 25, 36, 49, 64$ (from left to right).

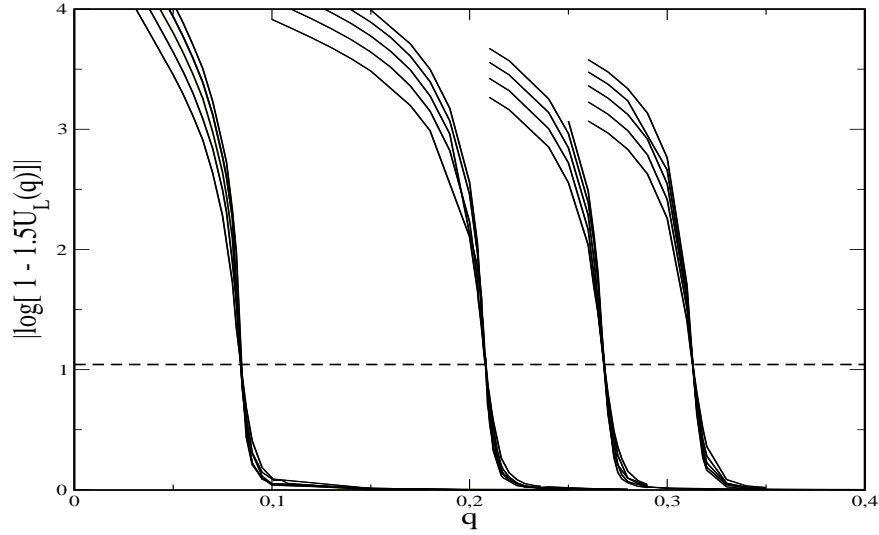


Figure 3. A set of fourth-order reduced Binder's cumulant as a function of q for $N_{PCS} = 4, 9, 16, 25$ (from left to right) and five values of lattice sizes L ($L = 100, 120, 140, 160$, and 180). The critical value $U^* = 0.606 \pm 0.004$ is shown by the horizontal line.

also increases. Despite the observed dependence between q_c and N_{PCS} , the value of the Binder's cumulant at the intersection point U^* does not depend on the size of the persuasive cluster: We obtained $U^* = 0.606 \pm 0.004$ (discontinuous horizontal line in Figure 3), for all N_{PCS} , which is in agreement with the result for the Ising model on the regular square lattice [12]. In order to construct the phase diagram for the block voter model, we have performed this analysis for several values of the parameter N_{PCS} .

The phase diagram in the $N_{PCS} \times q$ parameter space is shown in Figure 4. The curves were obtained using both the synchronous update with all adjacent spins to the

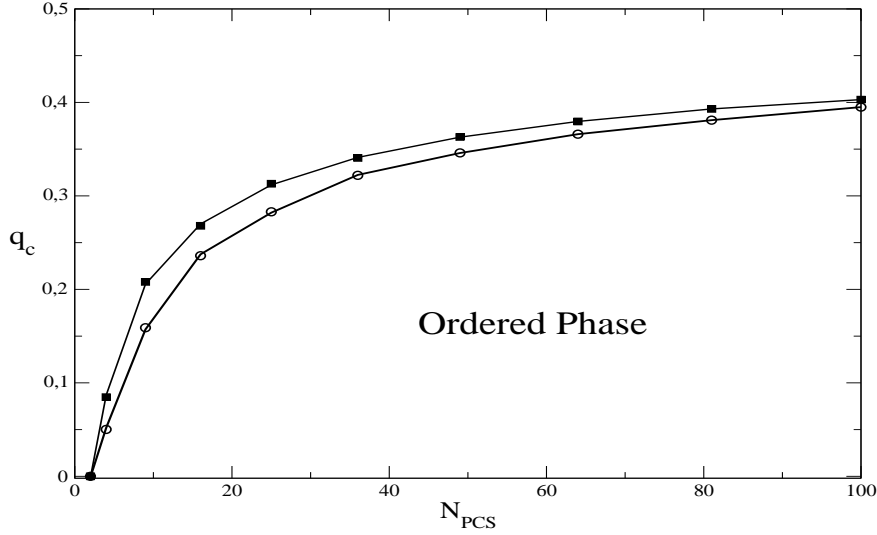


Figure 4. The phase diagram of the block voter model, showing the dependence of the critical noise parameter q_c on the size of persuasive cluster N_{PCS} , obtained from MC simulations using both synchronous (open symbols) and asynchronous (closed symbols) updates.

persuasive cluster being updated at the same time, and the asynchronous one when just a single spin randomly chosen in the neighbourhood of the PCS is updated. For both updates, there exists a phase transition only for values of $N_{PCS} > 2$. Moreover, the critical noise parameter q_c is an increasing function of the number of spins in the persuasive cluster. In other words, we can conclude that how large the PCS is more power of persuasion it has.

We turn now to the finite-size scaling (FSS) theory [13, 14] that allows us to extrapolate the information available from finite-system simulations to the relevant one in the thermodynamic limit. The critical behaviour of the block voter model is given by:

$$M_L(q) \sim L^{-\beta/\nu} \widetilde{M}(\varepsilon L^{1/\nu}), \quad (5)$$

$$\chi_L(q) \sim L^{\gamma/\nu} \widetilde{\chi}(\varepsilon L^{1/\nu}), \quad (6)$$

$$U_L(q) \sim \widetilde{U}(\varepsilon L^{1/\nu}), \quad (7)$$

where $\varepsilon = (q - q_c)$ is the distance from the critical noise. The exponents β/ν , γ/ν , and ν are, respectively, associated to the decay of the order parameter $M_L(q)$, the divergence of the susceptibility $\chi_L(q)$, and of the correlation length ($\xi \sim \varepsilon^{-\nu}$). The universal scaling functions $\widetilde{M}(\varepsilon L^{1/\nu})$ and $\widetilde{\chi}(\varepsilon L^{1/\nu})$ depend only on the scaling variable $x = \varepsilon L^{1/\nu}$.

The correlation length exponent ν can be obtained from the derivative of the Binder's fourth-order cumulant with respect to the noise parameter, namely, $U'_L(q_c)$. Moreover, we can check our estimates for the critical exponents considering the hyperscaling relation derived from the Rushbrooke and Josephson scaling laws, namely:

$$2\beta/\nu + \gamma/\nu = d, \quad (8)$$

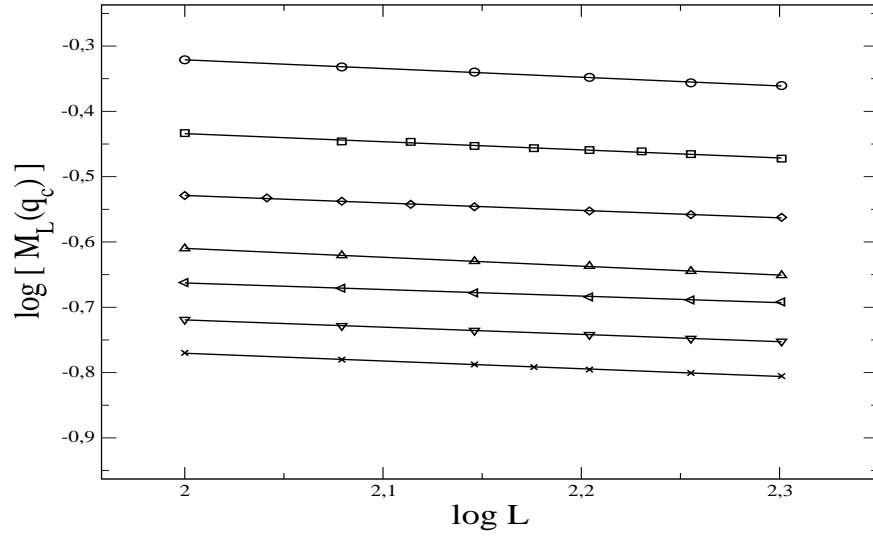


Figure 5. Log-log plot of the size dependence of $M_L(q_c)$ for PCS sizes $N_{PCS} = 4, 9, 16, 25, 36, 49$, and 64 from top to bottom. The slopes of the solid lines, obtained from the best linear fits to the data points, yield estimates for the exponent β/ν in good agreement with the exact value 0.125 .

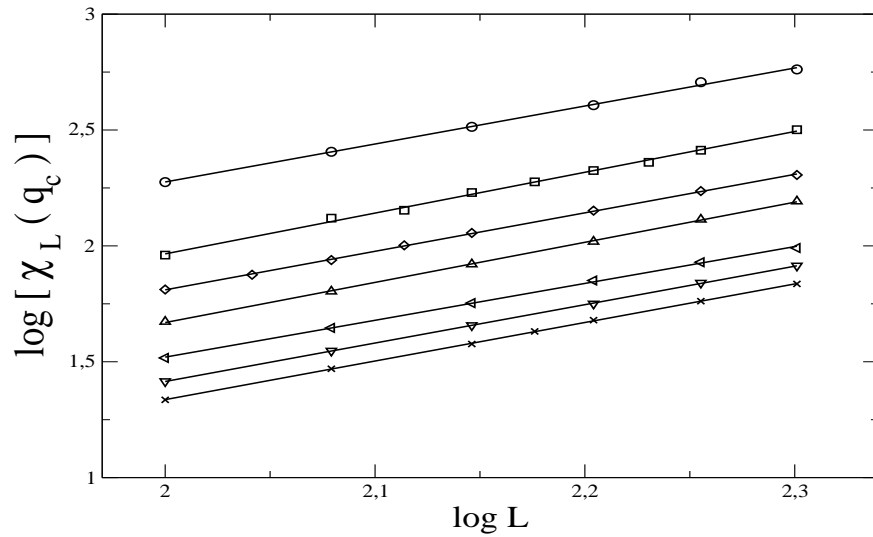


Figure 6. The same as in Fig. 5, but for the dependence of $\chi_L(q_c)$. The slopes of the straight lines give the exponent γ/ν close to the exact value 1.75 .

Table 1. Results for the critical noise q_c , the critical exponents $\beta/\nu, \gamma/\nu, 1/\nu$ and the effective dimensionality D_{eff} for the Block Voter Model on regular square lattices considering different sizes of persuasive cluster spin N_{PCS} .

N_{PCS}	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}^a
4	0.0846(2)	0.130(2)	1.72(4)	1.05(2)	1.98(4)
9	0.2080(1)	0.124(4)	1.76(3)	0.98(1)	2.01(4)
16	0.2680(1)	0.114(2)	1.73(1)	0.96(2)	1.95(2)
25	0.3130(4)	0.132(1)	1.74(1)	0.95(2)	2.00(1)
36	0.3406(3)	0.130(3)	1.70(3)	0.96(3)	1.96(4)
49	0.3630(1)	0.113(3)	1.70(2)	0.96(2)	1.96(3)
64	0.3790(2)	0.118(2)	1.71(4)	0.96(3)	1.95(4)

^a Obtained using $D_{eff} = 2\beta/\nu + \gamma/\nu$.

which is valid for Euclidean dimension d less than the upper critical dimension d_u [15].

Figures 5 and 6 show the dependence of the magnetization and susceptibility on system size L , at $q = q_c$, when different values of N_{PCS} are considered. From the scaling relations (equations 5) and (6), respectively, for a given N_{PCS} the corresponding straight lines in these log-log plots have slopes equal to the exponents β/ν and γ/ν . This analysis yields exponents very close to the two-dimensional Ising exponents, namely, $\beta/\nu = 0.125$ and $\gamma/\nu = 1.75$. Similarly, from the slopes of the resulting straight lines in the log-log plots of the derivative of the Binder's cumulant with respect to the noise parameter, at $q = q_c$, we determine the exponent associate with the correlation length in good agreement with the exact value $\nu = 1.0$. Finally, we have checked whether the calculated exponents satisfy the hyperscaling relation (equation 8) with $D_{eff} = 2\beta/\nu + \gamma/\nu$. For all PCS sizes considered we obtained $D_{eff} = 2$, within error bars. Our results are summarized in Table I.

According to Grinstein's conjecture [16], all equilibrium models with up-down symmetry on regular lattices have Ising-like critical behaviour. This conjecture has also been verified for several nonequilibrium model systems[17, 18, 19, 20]. Moreover, by considering the universality of the critical exponents for arbitrarily large but finite range of the interactions (see, e.g., [21, 22] and references therein), we actually should not expect even such a small dependence in the calculated exponents with the number of persuasive spins N_{PCS} . In fact, the critical behaviour of the present nonequilibrium Block Voter model and the equilibrium (finite-range interaction) Ising model are described by the same critical exponents.

In order to quantify the above statement, we shall consider a more accurate analysis that consists of obtaining the universal curves for the magnetization $\tilde{M}(x) = M_L(q)L^{\beta/\nu}$ and for the susceptibility $\tilde{\chi}(x) = \chi_L(q)L^{\gamma/\nu}$ with $x = (q - q_c)L^{1/\nu}$, which follows from equations 5 and 6, respectively, and represent the data collapsing for fixed N_{PCS} and different sizes L of the system. It is worth to note that the universal curves are obtained

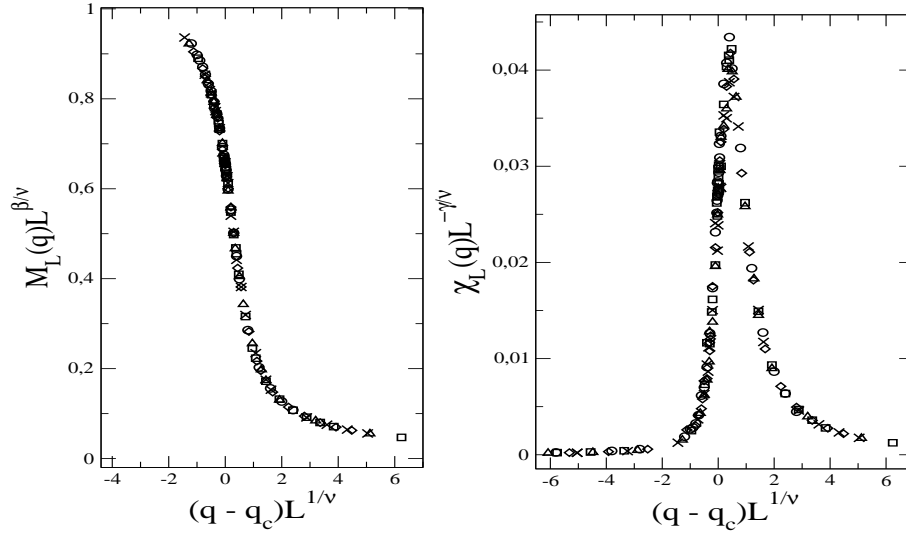


Figure 7. Data collapsing of the order parameter and of the susceptibility for five different values of $L = 100, 120, 140, 160$, and 180 with $N_{PCS} = 9$. The universal functions are consistent with Ising exponents: $\beta/\nu = 0.125$, $\gamma/\nu = 1.75$ and $1/\nu = 1.0$.

once the correct values for the exponents are used. Figure 7 shows the data collapse for the case of $N_{PCS} = 9$ and five different system sizes $L = 100, 120, 140, 160$, and 180 , using the following exponents: $\beta/\nu = 0.125$, $\gamma/\nu = 1.75$ and $1/\nu = 1.0$. A similar analysis applied to other values of N_{PCS} supports the conclusion that the present model is in the same universality class of the two-dimensional Ising model.

4. Conclusion

In this paper, Monte Carlo simulations and finite-size scaling theory was used to study a medium-ranged interactions version of voter model with outflow dynamics. The model is defined in terms of two parameters, the size of the persuasive cluster of spins (N_{PCS}) and the noise parameter q associated with the resistance that every spin has for accepting such outflow influence. Considering both synchronous and asynchronous updates, the resulting phase diagram in the $N_{PCS} \times q$ parameter space indicates that the region where there exists an ordered phase increases with increasing range of the interactions, meaning that how large the persuasive cluster is more power of persuasion it has. The calculated critical exponents for different sizes of PCS support the well-known criterion of universality class, stating that medium-range interactions models with up-down symmetry exhibit Ising-like critical behaviour for all arbitrarily large but finite range of the interactions.

Acknowledgments

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